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Bilinearization and soliton solutions of the $N = 1$ supersymmetric sine–Gordon equation

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Abstract

We obtain the bilinear form of the supersymmetric sine–Gordon equation. Through the Hirota approach we contruct multisoliton solutions for this equation. We find that, in contrast to the purely bosonic case, the solitons are 'dressed' through their mutual interaction and we compute this dressing explicitly. Using the Ablowitz, Ramani and Segur algorithm we verify that the supersymmetric sine–Gordon equation possesses the Painlevé property.

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Two of the most characteristic properties of integrable systems are the fact that their solutions are devoid of movable critical singularities (the Painlevé property) and the fact that they possess soliton solutions with an arbitrary number of solitons. The ubiquity of these properties in integrable evolution equations has elevated them to the rank of integrability criteria. Although the integrability of a given system is proven only when it is actually integrated (by inverse scattering methods, direct linearization or whatever other method is adequate), if a PDE possesses one of these two characteristic properties it can be considered a strong candidate for integrability (and even more so when both properties are present). Efficient methods have been proposed for their examination.

Singularity analysis, which goes back to the works of 19th century mathematicians, was adapted for use in evolution equations by Ablowitz, Ramani and Segur (ARS) in the 1970s [1]. Loosely stated, the ARS conjecture claims that all evolution equations integrable through inverse scattering methods possess the Painlevé property. The ARS method was cast in a form more convenient for PDEs through the formulation of Weiss *et al* [2]. In its current practical application singularity analysis examines the solution of a given PDE around some manifold $\chi = 0$ and investigates the possible existence of multivaluedness. If the solution is single valued for every noncharacteristic singularity manifold then the equation is proclaimed to possess the Painlevé property. One substantial simplification came from the Kruskal ansatz [3], which makes the assumption that the singularity manifold is solved for one of the variables (e.g. x). The fact that $\chi_x = 1$ greatly alleviates the calculations.

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On the other hand the construction of soliton solutions for a given nonlinear evolution equation is greatly simplified if one uses the Hirota bilinear formalism [4]. Hirota starts from a nonlinear equation and by introducing a dependent-variable transformation writes it in bilinear form. Once the latter is obtained, the construction of the soliton solution is straightforward: it is given by a finite sum of exponentials.

The study of a given nonlinear equation using the two methods mentioned above is interesting also when these methods are not used as integrability detectors. The Painlevé analysis gives precise information on the structure of the singularities of the equation, a fact which allows in some cases the derivation of auto-Bäcklund transformations and even of Lax pairs [5]. The Hirota approach on the other hand allows the explicit construction of the part of the solution which shows a coherent behaviour. While the solutions thus obtained are not the most general ones (even with the assumption of fast decay at infinity) they give a precise idea of the dynamics of the evolution.

In this paper we are going to examine the supersymmetric sine–Gordon equation from the point of view of the two integrability criteria just mentioned. The supersymmetric version of the sine–Gordon equation was introduced in [6] from purely physical motivations. The inverse problem for this system was obtained in [7] and used in [8] to obtain (among other things) the one-soliton solution. Before proceeding to the study of the sine–Gordon equation let us give a brief reminder of the notations used.

Our starting point is a standard (bosonic) nonlinear equation for a variable $u(x,t)$ which is a commuting field. To this component we add two fermionic variables $\phi(x,t)$ and $\psi(x,t)$, which are anticommuting of Grassmann type. In fact we shall obtain a nonlinear partial differential equation for the variables $u(x,t)$, $\phi(x,t)$ and $\psi(x,t)$, these functions taking values in the even and odd sectors of an infinite-dimensional Grassmann algebra [9]. One important remark: we are not interested in just *any* fermionic extension of the nonlinear evolution equations but only in those which are invariant under some supersymmetry transformations. For the proper mathematical formulation we need to extend the classical spacetime (x,t) to a super-spacetime (x,t,ξ,θ) where ξ and θ are Grassmann variables (and thus $\xi^2=0, \theta^2=0$). Simultaneously we extend the fields (u,ϕ,ψ) to a larger one $\Phi(x,t,\xi,\theta)$, which can be either bosonic or fermionic. For the extension of the sine–Gordon equation the proper choice, leading to a nontrivial result, is a bosonic field $\Phi = u/2+\xi\phi+\theta\psi+\xi\theta F$ (and it turns out that, for the sine–Gordon equation, $F=-\sin(u/2)$). In the case at hand the only supersymmetric invariance we demand is space and time invariance associated with the transformations: $x\to x-\kappa\xi$, $\xi\to\xi+\kappa$ and $t\to t-\lambda\theta$, $\theta\to\theta+\lambda$ (where κ and λ are anticommuting constants). These transformations are generated by the operators $Q_x=\partial_\xi-\xi\partial_x$ and $Q_t=\partial_\theta-\theta\partial_t$, which anticommute with the covariant derivatives $\mathcal{D}_x=\partial_\xi+\xi\partial_x$ and $\mathcal{D}_t=\partial_\theta+\theta\partial_x$ respectively. (Notice that $\mathcal{D}_x^2=\partial_x$, $\mathcal{D}_t^2=\partial_t$ and $\mathcal{D}_x\mathcal{D}_t=-\mathcal{D}_t\mathcal{D}_x$.) Expressions written in terms of the covariant derivatives and the superfield Φ are manifestly supersymmetric invariant.

Using these notations we can write the supersymmetric sine–Gordon equation as

$$\mathcal{D}_x\mathcal{D}_t\Phi=\sin\Phi. \tag{1}$$

Equation (1) can be equivalently written in components as

$$\begin{aligned}
&\partial_x\partial_t u=-\sin u+2\sin\frac{u}{2}\phi\psi\\
&\partial_t\phi=-\cos\frac{u}{2}\psi\\
&\partial_x\psi=\cos\frac{u}{2}\phi.
\end{aligned} \tag{2}$$

When the two fermionic fields ϕ,ψ vanish one readily recovers the standard sine–Gordon equation.

In order to construct the soliton solutions of (1) we start by bilinearizing it with the help of the super-Hirota operator we introduced in [10]. Its form, defined by its action on a pair of Grassmann-valued functions (f, g), is the following:

$$S_x f \cdot g := (\mathcal{D}_x f)g - (-1)^{|f|} f(\mathcal{D}_x g) \tag{3}$$

where $\mathcal{D}_x$ is the covariant derivative and $|f|$ is the Grassmann parity of the function f, which is zero if the function is bosonic and unity if the function is fermionic. The operator S can be thought of in some sense as the 'square root' of the Hirota operator D. We have in fact

$$S^{2N} f \cdot g = D^N f \cdot g. \tag{4}$$

In the case of the sine–Gordon equation we shall use a slight variant [10] of this operator acting on bosonic functions. We define S_{xt} as

$$S_{xt} f \cdot f := f \mathcal{D}_x \mathcal{D}_t f - (\mathcal{D}_x f)(\mathcal{D}_t f) \tag{5}$$

for any bosonic function f. One can easily see that this operator is gauge invariant with respect to the super-gauge

$$f \to f \mathrm{e}^{(kx+wt+\alpha\xi+\beta\theta)}$$

for any bosonic constants k, w and fermionic constants α and β.

Next we need a suitable transformation for the dependent variable ϕ. It turns out that the correct ansatz is

$$\Phi = \mathrm{i} \log \frac{F_-}{F_+} \tag{6}$$

where F_-, F_+ are Grassmann-valued τ-functions. Using for (6) in the sine–Gordon equation we obtain the following quadrilinear equation:

$$(S_{xt} F_- \cdot F_-) F_+^2 - F_-^2 S_{xt} F_+ \cdot F_+ = -\tfrac{1}{2}(F_-^2 - F_+^2) F_- F_+. \tag{7}$$

As usual, given the freedom due to the presence of *two* τ-functions we can split the equation into two parts:

$$S_{xt} F_- \cdot F_- = K F_-^2 - \tfrac{1}{2} F_- F_+ \tag{8a}$$

$$S_{xt} F_+ \cdot F_+ = K F_+^2 - \tfrac{1}{2} F_- F_+ \tag{8b}$$

where K is a free separation variable which can be put to zero through the gauge transformation $F_\pm \to \mathrm{e}^{K\theta\xi} F_\pm$. The bilinear form of the sine–Gordon equation thus becomes

$$S_{xt}(F_- \cdot F_- - F_+ \cdot F_+) = 0 \tag{9a}$$

$$S_{xt} F_+ \cdot F_+ + \tfrac{1}{2} F_- F_+ = 0. \tag{9b}$$

Starting from this bilinear form we can construct in a systematic way the soliton solutions of the supersymmetric sine–Gordon equation. Let us start with the one-soliton soltion. It turns out that

$$F_\pm = \mathrm{e}^{\xi\theta/2} \pm \mathrm{e}^{(kx+\omega t)} \mathrm{e}^{-(\xi+\zeta)(\theta+\lambda)/2} \tag{10}$$

is a solution of (9) provided the bosonic dispersion relation

$$\omega = -\frac{1}{k} \tag{11a}$$

and the fermionic one

$$\lambda = k\zeta \tag{11b}$$

are satisfied.

Having the one-soliton solution we proceed to construct the two-soliton one. We introduce $\eta = kx - t/k + \eta_0$ and find

$$F_\pm = e^{\xi\theta/2} \pm e^{\eta_1} e^{-(\xi+\zeta_1)(\theta+\lambda_1)/2} \pm e^{\eta_2} e^{-(\xi+\zeta_2)(\theta+\lambda_2)/2} + A_{12}\left(1 - \frac{k_1^2 + 3k_1k_2 + k_2^2}{2(k_1-k_2)}\zeta_1\zeta_2\right) e^{\eta_1+\eta_2} e^{(\xi+\alpha_{12}\zeta_1+\alpha_{21}\zeta_2)(\theta+\alpha_{12}\lambda_1+\alpha_{21}\lambda_2)/2} \tag{12}$$

where $\alpha_{ij} = (k_i+k_j)/(k_i-k_j)$ and $A_{ij} = (k_i-k_j)^2/(k_i+k_j)^2$, i.e. the classical interaction term which is corrected here by a term containing ζ_1 and ζ_2. Just as we noticed in the case of super-KdV [11], there exists a contribution which appears to be characteristic of supersymmetric systems, namely the one in the last exponentials involving α. This term corresponds, in fact, to a dressing of each soliton through its interaction with the other.

Having established the mechanism of dressing we can extend our construction to the higher-soliton solutions. We have for example for the three-soliton case

$$\begin{aligned}
F_\pm = {} & e^{\xi\theta/2} \pm e^{\eta_1} e^{-(\xi+\zeta_1)(\theta+\lambda_1)/2} \pm e^{\eta_2} e^{-(\xi+\zeta_2)(\theta+\lambda_2)/2} \pm e^{\eta_3} e^{-(\xi+\zeta_3)(\theta+\lambda_3)/2} \\
& + A_{12}\left(1 - \frac{k_1^2 + 3k_1k_2 + k_2^2}{2(k_1-k_2)}\zeta_1\zeta_2\right) e^{\eta_1+\eta_2} e^{(\xi+\alpha_{12}\zeta_1+\alpha_{21}\zeta_2)(\theta+\alpha_{12}\lambda_1+\alpha_{21}\lambda_2)/2} \\
& + A_{13}\left(1 - \frac{k_1^2 + 3k_1k_3 + k_3^2}{2(k_1-k_3)}\zeta_1\zeta_3\right) e^{\eta_1+\eta_3} e^{(\xi+\alpha_{13}\zeta_1+\alpha_{31}\zeta_3)(\theta+\alpha_{13}\lambda_1+\alpha_{31}\lambda_3)/2} \\
& + A_{23}\left(1 - \frac{k_2^2 + 3k_2k_3 + k_3^2}{2(k_2-k_3)}\zeta_2\zeta_3\right) e^{\eta_2+\eta_3} e^{(\xi+\alpha_{23}\zeta_2+\alpha_{32}\zeta_3)(\theta+\alpha_{23}\lambda_2+\alpha_{32}\lambda_3)/2} \\
& \pm A_{12}A_{13}A_{23}\left(1 - \frac{k_1^2 + 3k_1k_2 + k_2^2}{2(k_1-k_2)}\alpha_{13}\zeta_1\alpha_{23}\zeta_2\right) \\
& \times \left(1 - \frac{k_1^2 + 3k_1k_3 + k_3^2}{2(k_1-k_3)}\alpha_{12}\zeta_1\alpha_{32}\zeta_3\right) \\
& \times \left(1 - \frac{k_2^2 + 3k_2k_3 + k_3^2}{2(k_2-k_3)}\alpha_{21}\zeta_2\alpha_{31}\zeta_3\right) e^{\eta_1+\eta_2+\eta_3} \\
& \times e^{-(\xi+\alpha_{12}\alpha_{13}\zeta_1+\alpha_{21}\alpha_{23}\zeta_2+\alpha_{31}\alpha_{32}\zeta_3)(\theta+\alpha_{12}\alpha_{13}\lambda_1+\alpha_{21}\alpha_{23}\lambda_2+\alpha_{31}\alpha_{32}\lambda_3)/2}.
\end{aligned} \tag{13}$$

We remark that, as expected from the integrable character of the system, no new freedom enters at this stage, be it in the interaction or the dressing. However, the latter is most important and one must carefully account for the dressing of each soliton through it interaction with *every* other one. (The bulk of the calculations involved in the verification of the solution (13) requires the use of an algebraic manipulation program such as REDUCE).

We now turn to the singularity analysis [12] of the supersymmetric sine–Gordon equation. We start by rewriting in a rational form. We introduce $g = e^{iu/2}$ and obtain the system

$$gg_{xt} - g_x g_t + \tfrac{1}{4}(g^4 - 1) - \tfrac{1}{2}(g^3 - g)\phi\psi = 0 \tag{14a}$$

$$\psi_x - \frac{1}{2}\left(g + \frac{1}{g}\right)\phi = 0 \tag{14b}$$

$$\phi_t + \frac{1}{2}\left(g + \frac{1}{g}\right)\psi = 0. \tag{14c}$$

Next, we look for the leading singular behaviour of this system in the form $\phi = \phi_0\chi^\alpha$, $\psi = \psi_0\chi^\beta$, $g = g_0\chi^\gamma$, where $\chi(x,t) = 0$ is the singularity manifold and ϕ_0 and ψ_0 are fermionic while g_0 is bosonic. It turns out that the leading singularity corresponds to $\alpha = \beta = \gamma = -1$. Using the Kruskal ansatz we assume that $\chi_x = 1$, in which case we can consider that ϕ_0, ψ_0 and g_0 are functions solely of t. By balancing the appropriate terms in (14) we find that $g_0 = 2\mathrm{i}\sqrt{\chi_t}$ (up to a part which will involve a product of two fermionic functions) and $\psi_0 = -g_0\phi_0/2$. This last relation is of particular interest. Indeed, while the dominant singular terms in (14*a*) diverge like χ^{-4}, the product $g^3\phi\psi$ seems to lead to a χ^{-5} divergence. However since $\psi_0 \propto \phi_0$ the dominant term in the product $\phi\psi$ vanishes and we are left with the subdominant, χ^{-4} term, at the same order as the remaining ones.

The computation of the resonances follows the standard procedure. We put $\phi = \phi_0(1 + a\chi^\nu)/\chi$, $\psi = \psi_0(1 + b\chi^\nu)/\chi$ and $g = g_0(1 + c\chi^\nu)/\chi$ and obtain at lowest order a homogeneous system for the determination of a, b, c. From the vanishing of the determinant of this system we find $\nu = -1, 0, 2, 2$. Moreover, from the structure of the equations one expects one of the two $\nu = 2$ resonances to be associated with a bosonic integration constant while the other one is related to a fermionic one. To keep a long story short, we obtain the following expansion around the free singularity manifold $\chi = 0$:

$$
\begin{aligned}
\phi &= \frac{\phi_0}{\chi} + \phi_2\chi + \cdots \\
\psi &= -\mathrm{i}\frac{\phi_0\sqrt{\chi_t}}{\chi} + \mathrm{i}\frac{\phi_{0,t}}{\sqrt{\chi_t}} + \left(\mathrm{i}\phi_2\sqrt{\chi_t} + \mathrm{i}\frac{\phi_0\phi_{0,t}\phi_2}{2\sqrt{\chi_t}} - \mathrm{i}\frac{\phi_0}{4\sqrt{\chi_t}} + \frac{1}{2}\phi_0 g_2\right)\chi + \cdots \\
g &= \mathrm{i}\left(2\sqrt{\chi_t} + \frac{\phi_0\phi_{0,t}}{\sqrt{\chi_t}}\right)\frac{1}{\chi} + 2\mathrm{i}\sqrt{\chi_t}\phi_0\phi_2 + g_2\chi + \cdots
\end{aligned}
\qquad (15)
$$

where ϕ_0 and ϕ_2 are free fermionic functions of t (corresponding to the resonances $\nu = 0, 2$) and g_2 is a bosonic free function of t (which is the last integration constant associated with the second $\nu = 2$ resonance).

We remark that the solution of the supersymmetric sine–Gordon equation has an expansion which is indeed of Painlevé type. Only integer powers of χ appear in the expansion. The resonances' compatibility conditions at $\nu = 0$ and 2 are satisfied, introducing just the right number of integration constants and thus the solution is free of logarithmic terms. Thus the expansion around the movable singularity $\chi = 0$ is single valued and the equation has the Painlevé property.

Let us summarize our results here. In this paper we have presented the bilinearization of the supersymmetric sine–Gordon equation. This bilinearization was obtained using the supersymmetric analogue of the Hirota bilinear operator we introduced in [10]. Starting from this bilinear form we have constructed the soliton solutions. The important feature of these soliton solutions in the supersymmetric setting is that the solitons become dressed through the interaction with each other. The explicit calculation of the dressing makes possible the systematic contruction of multisoliton solutions. We have also verified, through the application of the ARS algorithm, that the supersymmetric sine–Gordon equation possesses the Painlevé property. Both approaches (bilinearization plus soliton construction and singularity analysis) can easily be transposed to other supersymmetric integrable equations.

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